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ON TREE CHARACTERIZATIONS OF G_δ-EMBEDDINGS AND SOME BANACH SPACES

BY

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ABSTRACT

We show that a one-to-one bounded linear operator T from a separable Banach space E to a Banach space X is a G_δ-embedding if and only if every T-null tree in S_E has a branch which is a boundedly complete basic sequence. We then consider the notions of regulators and skipped blocking decompositions of Banach spaces and show, in a fairly general set up, that the existence of a regulator is equivalent to that of special skipped blocking decomposition. As applications, the following results are obtained.

(a) A separable Banach space E has separable dual if and only if every w^*-null tree in S_{E^*} has a branch which is a boundedly complete basic sequence.

(b) A Banach space E with separable dual has the **point of continuity property** if and only if every w-null tree in S_E has a branch which is a boundedly complete basic sequence.

We also give examples to show that the tree hypothesis in both the cases above cannot be replaced in general with the assumption that every normalized w^*-null (w-null in (b)) sequence has a subsequence which is a boundedly complete basic sequence.

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1. Introduction

In [2] Bourgain and Rosenthal introduced the following notion of G_δ-embedding. A bounded linear one-to-one operator $T : E \to Y$ from a Banach space E into a Banach space Y is called a G_δ-embedding if the image $T(D)$ of every norm closed bounded and separable subset $D \subseteq E$ is a G_δ-set in Y. The usefulness of the notion of G_δ-embeddings in Banach space theory was illustrated by many authors, see, for example, [2], [4]–[9].

We will use standard Banach space theoretic notation (see [12]). For example, the unit ball and the the unit sphere of a Banach space E will be denoted by B_E and S_E respectively.

The following result (which we will use in the paper) was established in [8] by Ghoussoub and Maurey.

(A) *A bounded linear one-to-one operator $T : E \to Y$ from a Banach space E into a Banach space Y is a G_δ-embedding if and only if for any $\delta > 0$ and any δ-separated sequence $(y_n) \subseteq B_E$, (that is $\|y_i - y_j\| \geq \delta$, for $i \neq j$), the set $\{Ty_n\}_{n=1}^{\infty}$ is not dense in itself.*

Let us note that this characterization of G_δ-embeddings, as well as its definition is of the topological nature. However, it is possible to characterize G_δ-embeddings in geometrical (in the sense of Banach spaces geometry) terms. For that we need the following definition [4].

Let $T : X \to Y$ be linear bounded operator from a Banach space X into a Banach space Y. Denote by Σ the set of all finite ordered subsets of the unit sphere S_X of the space X. A function $\varepsilon_T : \Sigma \to \mathbb{R}^+$ is called a T-regulator for boundedly complete basic sequence (T-RBCBS, for short) if every sequence $(x_n) \subseteq S_X$ satisfying $\|Tx_{n+1}\| \leq \varepsilon_T(\{x_1, x_2, \ldots, x_n\})$, is a BCBS.

The following result which is also one of our main tools in this paper, was proved in [4].

(B) *Let $T : X \to Y$ be a linear bounded one-to-one operator from a separable Banach space X into a Banach space Y. Then T is a G_δ-embedding if and only if there is T-RBCBS.*

Clearly, (**B**) implies (see [7] and [9]):

(C) *If $T : X \to Y$ is a G_δ-embedding then any T-null sequence $\{x_n\} \subseteq S_X$, that is a sequence with the property that $\lim_n Tx_n = 0$, contains a subsequence which is a BCBS.*

However, this weaker property is not enough to characterize G_δ-embeddings (see Example 4.3 in Section 3).

We now consider a property which is formally intermediate between the ones described in **(B)** and **(C)**. First we recall the definition of a tree in a Banach space.

Let $\mathbb{N}^{<\omega}$ denote all non-empty finite ordered subsets of $\mathbb{N}$ in its natural partial order. A tree $(x_A)_{A\in\mathbb{N}^{<\omega}}$ in a Banach space E is a family of elements in E indexed by $\mathbb{N}^{<\omega}$. A sequence $\{x_{A_n}\}_{n\geq 1}$ is called a **branch** of the tree if $|A_1| = 1$, A_n is an initial segment of A_{n+1} and $A_{n+1} \setminus A_n$ is a singleton for any n.

A tree $(x_A)_{A\in\mathbb{N}^{<\omega}} \subseteq E$ is called T-null if every node sequence, that is the sequence $(x_{A\cup\{n\}})_{n=1}^{\infty}$, $A \in \mathbb{N}^{<\omega}$, is T-null.

Remark 1.1: If (x_n) is a sequence in E, then we can define an obvious tree by letting $x_A = x_{\max A}$. It is easy to see that the set of all branches of (x_A) thus defined, is the set of all subsequences of (x_n).

Various properties of trees and their branches in Banach spaces were studied in recent years (see, e.g., [15, 17]).

One of our main results is the following tree characterization of G_δ-embeddings.

THEOREM 1.2: *Let E be a separable Banach space and $T : E \to X$ a one-to-one bounded linear operator from E to a Banach space X. Then T is a G_δ-embedding if and only if every T-null tree in S_E has branch which is a BCBS.*

As we mentioned above the "tree-branch" assumption in this theorem cannot be replaced by the "sequence-subsequence" assumption (see Example 4.3 below).

Next we consider a more general set up for regulators. Let (P) be a property which a basic sequence in X may possess. We say (P) is stable if given a basic sequence $\{x_n\} \subset S_X$ with (P) and basis constant C, any sequence $\{y_n\} \subset X$ with $\sum_{n=1}^{\infty} \|x_n - y_n\| < 1/(2C)$, is a basic sequence with (P). For a subspace $\Gamma \subset X^*$ we denote $\mathcal{B}_\Gamma$ the family of all w-neighbourhoods of the origin in B_X generated by finite subsets $A \subset \Gamma$, i.e. the neighbourhoods of the form

$$V_A(\varepsilon) = \{x \in B_X : \ |f(x)| < \varepsilon, \ f \in A\}, \ \ A \subset \Gamma, \ \ \varepsilon > 0.$$

A map $W_P : \Sigma \to \mathcal{B}_\Gamma$ from the set of all finite ordered subsets of S_X into $\mathcal{B}_\Gamma$ is called a Γ-regulator of (P)-basic sequences if any normalized sequence $\{x_n\} \subset X$ with

$$x_{n+1} \in W_P(\{x_i\}_{i=1}^n), \quad n = 1, 2, \ldots,$$

is a (P)-basic sequence. For $\Gamma = X^*$ we write w-regulator instead of X^*-regulator.

Definition 1.3: Let X be a separable Banach space. For a sequence of subspaces (X_n) of X, with $X_i \cap X_j = \{0\}$, we denote by $X[k,l]$ the subspace $X_k \oplus X_{k+1} \oplus \cdots \oplus X_l$. A sequence of subspaces (X_n) is called a **complete minimal decomposition** (CMD, for short) if the following conditions are satisfied.
(i) $X = [X_n]_{n=1}^\infty$.
(ii) For each n, $X_n \cap [X_m]_{m \neq n} = \{0\}$.

A CMD (X_n) for X is called a **skipped blocking decomposition** (SBD,for short) if every skipped blocking of (X_n), that is, for sequences $n(k), m(k)$, such that $n(k) < m(k)+1 < n(k+1)$, $X[n(k), m(k)]_{k=1}^\infty$, is a Schauder decomposition for $[X[n(k), m(k)]]$.

We call a sequence (x_k), $x_k \in X[n(k), m(k)]$, $n(k) < m(k) + 1 < n(k + 1)$, $k = 1, 2, \ldots$ a **skipped block sequence** with respect to (X_n). A SBD (X_n) is said to be a (P)-SBD if every skipped block sequence is (P)-basic.

If in a SBD (X_k), each X_k is finite dimensional, we call the SBD a **skipped blocking finite dimensional decomposition** (SBFDD, for short).

Our next result in Section 2 shows that in a sense the notions of regulator and SBFDD are equivalent. However, in some cases (e.g., when dealing with trees) it is more convenient to operate with a regulator.

Proposition 1.4: *A separable Banach space E admits a w-regulator of (P)-basic sequences if and only if it has a (P)-SBFDD.*

We now talk about the applications of Theorem 1.2 and Proposition 1.4. There are two classes of Banach spaces which admit compact G_δ-embeddings.

The first is the class of all separable dual spaces. Recall that a one-to-one linear bounded operator $T : E \to X$ from a Banach space E into a Banach space X is called a semi-embedding if $T(B_E)$ is closed in X. It is trivial that any separable dual admits a compact semi-embedding and a semi-embedding defined on a separable space is a G_δ-embedding. By using Theorem 1.2 we prove that for a separable Banach space E the dual E^* is separable if and only if E^* has the following property:

(t^*) *any w^*-null tree in S_{E^*} has a branch which is a BCBS.*

The second is the class of separable spaces with the (PC) property. Recall that a Banach space E has the **point of continuity property** ((PC)-property, for short) if for every weakly closed bounded set $A \subseteq E$, the identity map from (A, w) to $(A, \|\cdot\|)$ has a point of continuity. The interrelation of G_δ-embeddings and the (PC) property is contained in the following result (see [8]).

(D) *A separable Banach space X has the (PC) property if and only if it admits a compact G_δ-embedding $T : X \to Y$ into some Banach space Y.*

In fact, Y may be taken to be ℓ_2. Clearly, if $w - \lim_n x_n = 0$, $x_n \in X$, $n = 1, 2, ...,$ then $\lim_n \|Tx_n\| = 0$.

Note that by combining Theorem 1.2 with **(D)** we can immediately get that a Banach space E with separable dual has the (PC)-property if and only if

(t) *any w-null tree in S_E has a branch which is a BCBS.*

We, however, give a characterization of separable Banach spaces with the (PC)-property without the restriction that the dual is separable. And here it is convenient to use regulators and Proposition 1.4.

We note also that Proposition 1.4 combined with (**B**) and **(D)** provides an alternative proof of the following result in [1, 8].

(E) *A separable Banach space has the (PC) property if and only if it admits a boundedly complete skipped blocking finite dimensional decomposition.*

As mentioned above, for any sequence (x_n) in E we can define an obvious tree by letting $x_A = x_{\max A}$. In [14] the following property of separable infinite-dimensional dual Banach spaces has been established.

($\mathbf{s}^*$) *Any w^*-null normalized sequence of functionals has a boundedly complete basic subsequence (BCBS).*

In [8], it was shown that in a separable Banach space with the (PC)-property, the following holds.

($\mathbf{s}$) *Any w-null normalized sequence has a BCBS.*

Of course both of the above results follow from the tree characterizations of spaces with separable duals and (PC)-spaces. However, it is natural to ask if property (s^*) (property (s)) alone characterizes Banach spaces with separable duals ((PC)-property). In Section 3 we give two examples to show that the in both the cases (t) and (t^*) cannot be replaced by (s) and (s^*). In particular, we show that that the (non-separable) dual JT^* of the celebrated (separable) James Tree space JT, introduced in [13] and later studied by Lindenstrauss and Stegal in [16], has property (s^*). Coming to (s), we prove that the space B_∞ constructed in [8], has property (s). Recall that B_∞^* is separable and B_∞ does not have the (PC)-property.

2. G_δ-embeddings; Regulators and SBFDD

The proof of the following simple lemma is standard and we omit it.

LEMMA 2.1: *Let X be a Banach space and $\Gamma \subset X^*$. Suppose X has a Γ-regulator for (P)-basic sequences. Then every Γ-null tree $(x_A)_{A\in[\mathbb{N}]^{<\omega}}$ in S_X has a branch which is a (P)-basic sequence.*

The following is one of our main results.

THEOREM 2.2: *Let E be a separable Banach space and $T : E \to X$ a one-to-one bounded linear operator from E to a Banach space X. The following assertions are equivalent.*

(a) *T is a G_δ-embedding;*
(b) *there exists a T-RBCBS;*
(c) *every T-null tree in S_E has a branch which is a BCBS.*

Proof. (a) $\Leftrightarrow$ (b) is proved in [4].

(b) $\Rightarrow$ (c) Let ε_T be a T-regulator. Let (x_A) be a T-null tree in S_E. Since $(x_{1,n})$ is a T-null sequence, there exists n_1 such that $\|Tx_{1,n_1}\| < \varepsilon_T(\{x_1\})$. Consider now the sequence $(x_{1,n_1,n})_n$ which is T-null. Hence there exists n_2 such that $\|Tx_{1,n_1,n_2}\| < \varepsilon_T(\{x_1, x_{1,n_1}\})$. Continuing, we get the desired branch.

(c)$\Rightarrow$ (a) Suppose to the contrary, T is not a G_δ-embedding. By [8, Theorem 1.2], there exists $\delta > 0$ and a δ-separated sequence $(y_n) \subseteq B_E$, (that is $\|y_i - y_j\| \geq \delta$, for $i \neq j$) such that (Ty_n) is dense in itself.

We first construct 2 trees: $(z_A)_{A\in\mathbb{N}^{<\omega}}$ in B_E and a T-null tree $\gamma = (u_A)_{A\in\mathbb{N}^{<\omega}}$ in S_E such that

(i) the image $(Tz_a)_{a\in\beta}$ of each branch $(z_a)_{a\in\beta}$, is dense in itself;
(ii) for any branch $\{u_{k_1,\dots,k_i}\}_{i=1}^\infty$ we have

$$\|Tu_{k_1,\dots,k_i}\| < 2^{-i}, \quad i = 1, 2, \dots; \tag{1}$$

(iii) any branch $\{z_{k_1,\dots,k_i}\}_{i=1}^\infty$ is δ-separated sequence in B_E;

(iv) for any branch $\{z_{k_1,\dots,k_i}\}_{i=1}^\infty$ we have

$$\operatorname{span}\{y_1, z_{k_1,\dots,k_i}\}_{i=1}^\infty = \operatorname{span}\{y_1, u_{k_1,\dots,k_i}\}_{i=1}^\infty. \tag{2}$$

Denote by $|A|$ the cardinality of the (finite) set A. We define the elements z_A and u_A by the induction on $n = |A|$. The elements z_A will be chosen from $\{y_i\}$.

Since (Ty_i) is dense in itself, we can find a subsequence $(y_{n_k}) \subset (y_i)$ such that

$$0 < \|Ty_1 - Ty_{n_k}\| \le \delta 2^{-k-1}, \quad k = 1, 2, \dots.$$

Put

$$z_k = y_{n_k}, \ \ u_k = \frac{z_k - y_1}{\|z_k - y_1\|}, \quad k = 1, 2, \dots,$$

that is, we defined z_A and u_A for $|A| = 1$. Next assume that we already defined z_B and u_B for $|B| \le n$, and define z_A and u_A for $|A| = n+1$. Write $n+1 = 2^l + j$, $j = 0, \dots, 2^l - 1$. Let B be an initial segment of A with $|B| = j$. Since $\{Ty_i\}$ is dense in itself, we can find a subsequence $(y_{m_k}) \subset (y_i)$, $y_{m_k} \neq z_C$, for every initial segment C of A, and for any k, and such that

$$0 < \|Tz_B - Ty_{m_k}\| \le \delta 2^{-k-n-1} \quad k = 1, 2, \dots.$$

Now, if t is the last element of A we put

$$z_A = y_{m_t}, \ \ u_A = \frac{z_A - z_B}{\|z_A - z_B\|}.$$

A straightforward verification shows that (i)–(iv) are satisfied.

By our assumption in (c), γ has a branch $\beta = \{u_{k_1,\dots,k_n}\}_{n=1}^\infty$ which is a BCBS. Put

$$x_n = u_{k_1,\dots,k_n}, \ v_n = z_{k_1,\dots,k_n}, \ n = 1, 2, \dots, \ x_0 = y_1, \ Y = [x_n]_{n=0}^\infty.$$

If $x_0 \in [x_n]_{n=1}^\infty$ then $\{x_n\}_{n=1}^\infty$ is a boundedly complete basis of Y. If $x_0 \notin [x_n]_{n=1}^\infty$ then $\{x_n\}_{n=0}^\infty$ is a boundedly complete basis of Y. The next part of the arguments in both cases is the same. So we assume without loss of generality that $\{x_n\}_{n=1}^\infty$ is a (boundedly complete) basis of Y.

Claim: $T|_Y$ is a G_δ-embedding of Y into X.

We first show how to finish the proof with the help of the Claim, and then prove it. By (iii) and (iv) $\{v_n\} \subset B_Y$ is a δ-separated sequence. However, by (i) the image $\{Tv_n\}$ is dense in itself. This is a contradiction to $T|_Y$ is a G_δ-embedding (see [8, Theorem 1.2]). Thus (c) $\Rightarrow$ (a) is proved.

It remains to prove the claim. Let us introduce a new norm on Y by

$$|||y||| = \sup_n \left\{ \left\| \sum_1^n a_i x_i \right\| : \ y = \sum a_i x_i \right\}.$$

Clearly $||| \cdot |||$ is equivalent to the original norm. Denote by V the unit ball in the norm $||| \cdot |||$. We show that the image TV is closed in X.

Let $Tu_m \to v$ where $u_m \in V$. Writing $u_m = \sum a_i^m x_i$, without loss of generality, we assume that $\lim_m a_i^m = a_i$ for each i. It is straightforward to check that $\sup_n \| \sum_i^n a_i x_i \| \leq 1$. Since (x_n) is BCBS, $\sum a_i x_i$ converges to some $u \in V$. Given ε we choose n such that $\sum_{n+1}^{\infty} \|Tx_i\| < \varepsilon/16$. We then choose m such that for all $i = 1, \ldots, n$, $|a_i^m - a_i| < \varepsilon/(4n\|T\|)$. Thus we have,

$$\|Tu_m - Tu\| \leq \left\| \sum_1^n (a_i^m - a_i) Tx_i \right\| + \left\| \sum_{n+1}^{\infty} (a_i^m - a_i) Tx_i \right\| < \varepsilon.$$

Hence, $Tu_m \to Tu, \ u \in V, v = Tu$. Therefore, $v \in TV$. This shows $T|_Y$ is a semi-embedding of Y in X and since Y is separable, T is a G_δ-embedding. This proves the claim. The proof of the theorem is complete. ∎

Our next result shows the equivalence of existence of regulators and that of special skipped blocking decompositions in a fairly general set up.

PROPOSITION 2.3: *Let X be a separable Banach space and (P) some stable property a basic sequence may possess. The following assertions are equivalent.*

(a) *There is a separable subspace $\Gamma \subset X^*$ such that X has a Γ-regulator W_P for (P)-basic sequences;*
(b) *X has a w-regulator W_P for (P)-basic sequences.*
(c) *X has a (P)-SBFDD.*

Proof. (a) $\Rightarrow$ (b) is trivial.

(b) $\Rightarrow$ (c) For $\varepsilon > 0$ and $A = \{f_1, f_2, \ldots, f_n\}$ a finite subset of S_{X^*}, denote

$$V_A(\varepsilon) = \{x \in B_X : |f_i(x)| < \varepsilon, \ i = 1, 2, \ldots, k\}.$$

Fix a sequence

$$\{\varepsilon_k\}, \ 0 < \varepsilon_k < 1/6, \ k = 1, 2, \ldots, \ \prod_{k=1}^{\infty} (1 + \varepsilon_k) < 2^{1/4}, \ \sum_{k=1}^{\infty} \varepsilon_k < 1/4.$$

Let $(x_n) \subseteq S_X$ be a dense sequence.

Put $X_1 = [x_1]$. We can write $X = X_1 \oplus E_0$ where $E_0 = {D^1}^\top$ for some finite set $D^1 \subseteq S_{X^*}$. Let $T_1 = \{z_1^1, z_2^1, \ldots, z_{j_1}^1\}$ be a $\varepsilon_1/2$-net in S_{X_1}. For each $i = 1, \ldots, j_1$, find $\delta_i^1 > 0$ and finite subset $A_i^1 \subseteq S_{X^*}$, such that

$$W_P(\{z_i^1\}) = V_{A_i^1}(\delta_i^1), \quad i = 1, \ldots, j_1.$$

Let $F_1 \subseteq S_{X^*}$ be a finite $(1+\varepsilon_1)^{-1}$-norming set for X_1, that is,

$$\|x\| \le (1+\varepsilon_1)\sup\{f(x) : f \in F_1\}, \quad x \in X_1.$$

Now let $x_2 = u_2 + v_2$ where $u_2 \in X_1$ and $v_2 \in E_0$. If $v_2 \neq 0$ pick $y_2^* \in S_{X^*}$ with $y_2^*(v_2) \neq 0$. If $v_2 = 0$ just put $y_2^* = 0$. Put

$$B_1 = \Big(\bigcup_i A_i^1\Big) \cup D^1 \cup \{y_2^*\} \cup F_1, \quad E_1 = B_1^\top.$$

Let X_2 be a finite dimensional subspace of E_0 containing v_2 such that $E_0 = X_2 \oplus E_1$. Thus $X = X_1 \oplus X_2 \oplus E_1$.

Put $\eta_2 = \frac{1}{2}\min\{\varepsilon_2, \delta_i^1 : i = 1, \ldots, j_1\}$ and let $T_2 = \{z_1^2, z_2^2, \ldots, z_{j_2}^2\}$ be a η_2-net in $S_{X_1 \oplus X_2}$.

Next for each k, $1 \le k \le j_2$, and for each pair (i,k), $1 \le i \le j_1$, $1 \le k \le j_2$, find

$$\delta_k^2 > 0, \quad \delta_{ik}^2 > 0,$$

and finite subsets

$$A_k^2 \subset S_{X^*}, \quad A_{ik}^2 \subset S_{X^*},$$

such that

$$W_P(\{z_k^2\}) = V_{A_k^2}(\delta_k^2), \quad W_P(\{z_i^1, z_k^2\}) = V_{A_{ik}^2}(\delta_{ik}^2).$$

Let $F_2 \subseteq S_{X^*}$ be a finite $(1+\varepsilon_2)^{-1}$-norming set for $X_1 \oplus X_2$.

Write $x_3 = u_3 + v_3$ where $u_3 \in X_1 \oplus X_2$ and $v_3 \in E_1$. If $v_3 \neq 0$ pick $y_3^* \in S_{X^*}$ with $y_3^*(v_3) \neq 0$. If $v_3 = 0$ just put $y_3^* = 0$.

Put

$$B_2 = B_1 \cup \Big(\bigcup_i A_i^2\Big) \cup \Big(\bigcup_{ik} A_{ik}^2\Big) \cup \{y_3^*\} \cup F_2, \quad E_2 = B_2^\top.$$

Let X_3 be a finite dimensional subspace of E_1 containing v_3 such that $E_1 = X_3 \oplus E_2$. Thus $X = X_1 \oplus X_2 \oplus X_3 \oplus E_2$.

Proceeding inductively, we construct:

(1) a sequence of finite-dimensional subspaces $X_k \subset X$ and decreasing sequence of finite-codimensional subspaces $E_k \subset X$, $k = 1, 2, \ldots$, such that

(i) $x_n \in X_1 \oplus \cdots \oplus X_n,\ E_{n-1} = X_{n+1} \oplus E_n,\ X = X_1 \oplus \cdots \oplus X_n \oplus E_{n-1},\ n = 1, 2, \ldots,$

and

(2) an increasing sequence $B_k \subset S_{X^*},\ k = 1, 2, \ldots,$ of finite sets, and a sequence $T_n = \{z_i^n\}_{i=1}^{j_n} \subset S_{X_1 \oplus \cdots \oplus X_n},\ n = 1, 2, \ldots,$ such that

(ii) $E_k = B_k^{\top},\ \ k = 1, 2, \ldots;$

(iii) there is a subset $F_n \subset B_n$ which $(1 + \varepsilon_n)^{-1}$ norms $X_1 \oplus \cdots \oplus X_n,\ n = 1, 2, \ldots;$

(iv) for each $k \le n$ and each collection $(i_1, \ldots, i_k), 1 \le i_1 \le j_1, \ldots, 1 \le i_k \le j_k$, there exist $\delta^n_{i_1,\ldots,i_k} > 0$ and finite sets $A^n_{i_1,\ldots,i_k} \subseteq B_n$ such that

$$V_{A^n_{i_1,\ldots,i_k}}(\delta^n_{i_1,\ldots,i_k}) = W_P(\{z^1_{i_1}, \ldots, z^k_{i_k}\})$$

(v) T_n is a η_n-net in $S_{X_1 \oplus \cdots \oplus X_n}$ for

$$\eta_n = \frac{1}{2} \min\{\varepsilon_n, \delta^{n-1}_{i_1,\ldots,i_k},\ 1 \le i_1 \le j_1, \ldots, 1 \le i_k \le j_k,\ k \le n-1\}.$$

From (i) it follows that $X_n \cap [X_m]_{m \neq n} = \{0\}$, and $X = [X_n]_{n=1}^{\infty}$. Thus (X_n) is a CMD. Next we need to verify the SBD-condition of Definition 1.3. For simplicity we consider the case: $n(k) = m(k) = 2k,\ k = 1, 2, \ldots$. The verification for the general case is similar. So let $y_k \in S_{X_{2k}}$. We need to check that (y_k) is a (P)-basic sequence. By (v), there exists $z^{2k}_{i_k} \in T_{2k}$ such that $\|y_k - z^{2k}_{i_k}\| \le \eta_{2k}$, $k = 1, 2, \ldots$. In particular,

$$\sum_{k=1}^{\infty} \|y_k - z^{2k}_{i_k}\| < 1/4. \tag{3}$$

Since by (i), $X_{2k} \subseteq E_{2k-2}$ and by (ii), $E_{2k-2} = B^{\top}_{2k-2}$, it follows that $f(y_k) = 0$ for each $f \in B_{2k-2}$. Hence

$$|f(z^{2k}_{i_k})| \le \eta_{2k-2}, \quad f \in B_{2k-2},\ k = 1, 2, \ldots.$$

In particular,

$$|f(z^{2k}_{i_k})| \le \varepsilon_{2k-2}, \quad f \in F_{2k-2},\ k = 1, 2, \ldots. \tag{4}$$

$$|f(z^{2k}_{i_k})| < \delta^{2k-2}_{i_1,\ldots,i_{2k-2}}, \quad f \in A^{2k-2}_{i_1,\ldots,i_{2k-2}},\ k = 1, 2, \ldots. \tag{5}$$

We claim that $\{z^{2k}_{i_k}\}$ is a basic sequence with basis constant < 2. Fix a finite set of numbers $\{a_i\}_{k=1}^{n+1}$ and denote $S_n = \sum_{k=1}^{n} a_k z^{2k}_{i_k}$ and $S_{n+1} = \sum_{k=1}^{n+1} a_k z^{2k}_{i_k}$.

By using (iii) find $f \in F_n$ with $\|S_n\| \le (1+\varepsilon_n)|f(S_n)|$. Next we write

$$\begin{aligned}\|S_n\| &\le (1+\varepsilon_n)|f(S_{n+1}) - f(a_{n+1} z^{2n+2}_{i_{n+1}})| \le (1+\varepsilon_n)(\|S_{n+1}\| + \varepsilon_n |a_{n+1}|)\\ &\le (1+\varepsilon_n)(\|S_{n+1}\| + \varepsilon_n(\|S_{n+1}\| + \|S_n\|)).\end{aligned}$$

Therefore,

$$\|S_n\| \le \frac{(1+\varepsilon_n)^2}{1-\varepsilon_n(1+\varepsilon_n)}\|S_{n+1}\| \le (1+\varepsilon_n)^4\|S_{n+1}\|,$$

where we used that $\varepsilon_n \in (0, 1/6)$. Hence $\{z^{2k}_{i_k}\}$ is a basic sequence with basis constant $C \le \prod_{n=1}^{\infty}(1+\varepsilon_n)^4 \le 2$.

Next from (iv) and (5) it follows that $\{z^{2k}_{i_k}\}$ has property (P). Finally from (3) we conclude that $\{y_k\}$ is a (P)-basic sequence.

(c) $\Rightarrow$ (a) Let $(X_k)_{k\ge1}$ be a (P)-SBFDD for X. A standard argument shows that there is a constant $C \ge 1$ such that every skipped block sequence of $(X_k)_{k\ge1}$ is a (P)-basic sequence with basis constant at most C. Fix $\varepsilon_k > 0$, $\sum \varepsilon_k < 1/(2C)$.

We define a regulator $W_P(\{x_i\}_{i=1}^k)$ for (P)-basic sequences inductively. Start with $k = 1$. Fix $x \in S_X$. Put $p_0 = -1$ and find p_1 and $y \in S_{[X_i]_{p_0+2}^{p_1}}$ such that $x = y + z$ where $\|z\| < \varepsilon/2$. We take $Y_1 = [X_i]_{i\ge p_1+2}$. Then Y_1 is a finite codimensional subspace of X. Let $A_1 \subseteq S_{Y_1^\perp}$ be a finite $\varepsilon_1/4^2$-net in $S_{Y_1^\perp}$. We define

$$W_P(\{x_1\}) = V_{A_1}(\varepsilon_1/4).$$

This defines W on each one point subset of S_X.

Suppose W_P has been defined for all k-point subsets of S_X, $k \ge 1$ and $\{x_1, x_2, \ldots, x_{k+1}\} \subseteq S_X$ such that

$$W_P(\{x_1, x_2, \ldots, x_k\}) = V_{A_k}(\varepsilon_k/4^k),$$

where A_k is a finite $\varepsilon_k/4^{k+1}$-net in $S_{Y_k^\perp}$, $Y_k = [X_i]_{i\ge p_k+2}$.

If $x_{k+1} \notin W_P(\{x_1, x_2, \ldots, x_k\})$ we define

$$W_P(\{x_1, x_2, \ldots, x_{k+1}\}) = W_P(\{x_1, x_2, \ldots, x_k\}).$$

If $x_{k+1} \in W_P(\{x_1, x_2, \ldots, x_k\})$ then for all $f \in A_k$, $|f(x_{k+1})| < \varepsilon_k/4^k$. Since A_k is a $\varepsilon_k/4^{k+1}$-net in $S_{Y_k^\perp}$, we can write $x_{k+1} = s + t$ where $s \in S_{Y_k}$ and $\|t\| < \varepsilon_k/4^k$. Next we choose $p_{k+1} > p_k + 2$ and $y_{k+1} \in S_{[X_k]_{p_k+2}^{p_{k+1}}}$ such that $s = y_{k+1} + z$ where $\|z\| < \varepsilon_k/4^k$. Note that $\|x_{k+1} - y_{k+1}\| < \varepsilon_k/2^k$.

Put $Y_{k+1} = [X_k]_{k \geq p_{k+1}+2}$. Y_{k+1} is a finite codimensional subspace of X. Let $A_{k+1} \subseteq S_{Y_{k+1}^\perp}$ be a finite $\varepsilon_{k+1}/4^{k+2}$-net in $S_{Y_{k+1}^\perp}$. We define

$$W_P(\{x_1, x_2, \dots, x_{k+1}\}) = V_{A_{k+1}}\Big(\frac{\varepsilon_{k+1}}{4^{k+1}}\Big).$$

Put $\Gamma = \operatorname{cl}\operatorname{span} \bigcup_k A_k$ and we check that W_P is a regulator for (P)-basic sequences. To this end, let $(x_k) \subseteq S_X$ satisfy $x_{k+1} \in W_P(\{x_1, x_2, \dots, x_k\})$. For each k, there exists $y_k \in S_{[X_j]_{p_{k-1}+2}^{p_k}}$ such that $\|x_k - y_k\| < \varepsilon_k/2^k$. Hence $\sum \|x_k - y_k\| < \sum \varepsilon_k/2^k < 1/(2C)$. By the construction (y_k) is a skipped block sequence of $(X_k)_{k\geq 1}$, and hence (y_k) is a (P)-basic sequence of basis constant at most C. By the stability of (P) it follows that (x_k) is (P)-basic sequence as well. This completes the proof. ∎

3. Applications.

The first application of Theorem 2.2 is a characterization of separable dual spaces among duals of separable spaces. We start with a proposition which is essentially proved in [4].

PROPOSITION 3.1: *Let a Banach space E admit a compact semi-embedding $T : E \to X$ which is a G_δ-embedding. Then E is isometric to a separable dual space.*

Proof. Let $K = T(B_E)$. Then K is a compact symmetric convex set. Let $Y = A_0(K)$ be the space of all affine continuous functions on K vanishing at the origin with sup-norm. A standard argument shows that E is isometric to Y^*. The rest of the proof runs along the lines of the proof of Theorem 3, (2)⇒ (3), in [4]. The proof is complete. ∎

THEOREM 3.2: *Let E be a separable Banach space. The following assertions are equivalent.*

(a) *E^* is separable.*
(b) *There exists a w^*-RBCBS for E^*.*
(c) *E^* has property (t^*).*

Proof. (a) ⇔ (b) is proved in [4, Theorem 3].

(b) ⇒ (c) follows from Lemma 2.1.

(c) ⇒ (a) Let $A : \ell_2 \to E$ be a compact operator with dense range. Denote $T = A^* : E^* \to \ell_2$. It is clear that T is a compact semi-embedding and every normalized T-null tree is w^*-null. Therefore from Theorem 2.2, $(c) \Rightarrow (a)$, it follows that T is a G_δ-embedding of E^* into ℓ_2 (recall that the notion of G_δ-embedding is separably defined). Apply Proposition 3.1 to complete the proof. ∎

We now consider (PC)-spaces. The following theorem provides a characterization of separable Banach spaces with the (PC) property in terms of the trees in the unit sphere. This also provides an alternative proof of the result (**E**) from the introduction.

THEOREM 3.3: *Let E be a separable Banach space. The following assertions are equivalent.*

(a) *E has the (PC)-property.*
(b) *There exists a separable subspace $\Gamma \subseteq E^*$ such that E has a W_Γ-RBCBS.*
(c) *E has a BCSBFDD $(X_k)_{k\geq 1}$.*
(d) *There exists a separable subspace $\Gamma \subseteq E^*$ such that every Γ-null tree in S_E has a branch which is BCBS.*

Proof. (a) ⇔ (b) is formulated in the short note [5] without proof. We give a proof here for the sake of completeness.

(a) ⇒ (b) Since E has the (PC)-property it follows [8] that there is a compact G_δ-embedding $T : E \to l_2$. We take $\Gamma = T^* l_2$. Then Γ is separable. By [4] (see Theorem 2.2, (a)⇔ (b)), there exists a T-RBCBS ε_T on E.

We define a W_Γ-RBCBS as follows. Let $x \in S_E$. Since $T^* B_{X^*}$ is compact there is a finite $\varepsilon_T(\{x\})/2$-net $\{f_i^1\}_{i=1}^{n_1}$ in $T^* B_{X^*}$. Define

$$W_\Gamma(\{x\}) = \{y \in B_E : |f_i^1(y)| < \varepsilon_T(\{x\})/2,\ i = 1, \ldots, n_1\}.$$

It is easily seen that for any $y \in W_\Gamma(\{x\})$, we have $\|Ty\| < \varepsilon_T(\{x\})$.

Now let $x_1, x_2 \in S_E$. By the compactness of $T^* B_{X^*}$ there is a $\varepsilon_T(\{x_1, x_2\})/2$-net $\{f_i^2\}_{i=1}^{n_2}$ in $T^* B_{X^*}$. Define

$$W_\Gamma(\{x_1, x_2\}) = \{y \in B_E : |f_i^2(y)| < \varepsilon_T(\{x_1, x_2\})/2,\ i = 1, \ldots, n_2\}.$$

The same argument as above shows that if $y \in W_\Gamma(\{x_1, x_2\})$, then $\|Ty\| < \varepsilon_T(\{x_1, x_2\})$.

Continuing, we get the desired regulator W_Γ.

(b) $\Rightarrow$ (a) Let $\Gamma \subset E^*$ be a separable subspace and W_Γ be a RBCBS. It is easily seen that Γ is total. Let $K \subset \Gamma$ be a norm-compact convex symmetric subset such that $\mathrm{cl\,span} K = \Gamma$. Without loss of generality we can assume that for $\{x_i\}_1^n \subseteq S_E$, there exists $\varepsilon = \varepsilon(\{x_i\}_i^n)$ such that

$$W_\Gamma(\{x_i\}_{i=1}^n) = \{x \in B_E : \ |f_j(x)| < \varepsilon, \ f_j \in K, \ j = 1, \ldots, m\}, \ \ \{x_i\}_{i=1}^n \subset S_E.$$

Next define a new norm on the space E as follows

$$|||x||| = \max\{|g(x)| : \ g \in K\}.$$

Let X be the completion of the space E with norm $||| \cdot |||$, and $T : E \to X$ be a natural (one-to-one) embedding. Clearly, $T^*(B_{X^*}) = K$. Now we define a map $\varepsilon_T(\{x_i\}_{i=1}^n) = \varepsilon$, where ε has come from (6). It is not difficult to see that ε_T is a T-RBCBS. By [4] (see Theorem 2.2, (b) $\Leftrightarrow$ (a)) we conclude that T is a G_δ-embedding. Therefore, E admits a compact G_δ-embedding. By [8] E has the (PC)-property.

The equivalence of (b) and (c) follows from Proposition 2.3.

(b) $\Rightarrow$ (d) follows from Lemma 2.1.

(d) $\Rightarrow$(a) We start as in the beginning of the proof of (b) $\Rightarrow$ (a), and construct the space X and the compact one-to-one operator $T : E \to X$ such that $\mathrm{cl\,span} T^*(X^*) = \Gamma$, (the condition (d) guarantees the totality of Γ). Form (d) it follows that any T-null tree has a branch which is BCBS. By Theorem 2.2, $(c) \Rightarrow (a)$, we get that T is a G_δ-embedding. By the result from [8] mentioned in the introduction we conclude that E has the (PC)-property. ∎

Let $(X_k)_{k\geq 1}$ be a CMD for a Banach space X. $(X_k)_{k\geq 1}$ is said to be **shrinking** if for every $f \in X^*$, $\lim_n \|f|_{[X_k]_{k\geq n}} 0\| = 0$. By [19, Theorem 4.1], X has a shrinking SBFDD if and only if X^* is separable. Also, it was shown in [19] that a SBFDD, which is skipped-shrinking, meaning that every skipped block sequence generates a shrinking basic sequence, is in fact, a shrinking SBFDD.

Corollary 3.4: *Let E^* be separable. The following assertions are equivalent.*

(a) *E has the (PC)-property.*
(b) *E has a w-RBCBS.*
(c) *E has a w-regulator for shrinking boundedly complete basic sequences.*
(d) *E has a shrinking BCSBFDD $(X_k)_{k\geq 1}$.*
(e) *E has property (t).*

Proof. Taking $\Gamma = X^*$, in Theorem 3.3, (a) $\Rightarrow$(b) follows.

(b) $\Rightarrow$(c) Let W be a w-RBCBS. We define a regulator W_1 for boundedly complete shrinking basic sequences.

Let (f_n) be a dense sequence in S_{E^*}. Take $\varepsilon_n \downarrow 0$. Define $W_1 : \Sigma \to \mathcal{B}$ as $W_1(\{x_1, x_2, \dots, x_n\}) = W(\{x_1, x_2, \dots, x_n\}) \cap V_{f_1, f_2, \dots, f_n}(\varepsilon_n)$. If $(x_n) \subseteq S_E$ satisfies $x_{n+1} \in W_1(\{\{x_1, x_2, \dots, x_n\})$ then $x_{n+1} \in W(\{x_1, x_2, \dots, x_n\})$ and hence it is BCBS. Also $x_{n+1} \in V_{f_1, f_2, \dots, f_n}(\varepsilon_n)$. By the density of (f_n) it follows that for every $f \in E^*$, $\|f\|_{[x_k]_{k \geq n+1}} \to 0$. Thus (x_n) is shrinking.

(c) $\Rightarrow$(d) By Proposition 2.3 it follows that E has a skipped shrinking BCS-BFDD $(X_k)_{k\geq 1}$, that is, every skipped block sequence of $(X_k)_{k\geq 1}$ generates a shrinking and BCBS. It was noted in [19] that an SBFDD, which is skipped-shrinking, is in fact, a shrinking SBFDD.

(d) $\Rightarrow$(e) follows from Lemma 2.1; (e) $\Rightarrow$(a) follows from Theorem 3.3. ∎

Remark 3.5: By [3, 19] the Banach spaces with the (PC)-property and separable dual form the class of Banach spaces such that the unit ball is Polish in the weak topology. Thus Corollary 3.4 gives a tree characterization of this class of Banach spaces among the spaces with separable dual.

4. Property (s) and (s^*)

In this section we prove that the properties (t) and (t^*) are indeed stronger than properties (s) and (s^*) and the tree hypothesis in Theorem 2.2, Theorem 3.2, and Corollary 3.4 cannot be in general weakened to the sequence hypothesis.

We start with general results which we then apply to the space JT.

THEOREM 4.1: *Let X be a separable Banach space satisfying the following conditions:*

(a) *X has the (PC)-property.*
(b) *X^* is separable.*
(c) *X^{**}/X is reflexive.*

Then X^{**} has property (s^*).

The proof of the following lemma follows from a standard argument.

LEMMA 4.2: *Let L be a Banach space and $E \subseteq L$ such that $E = [X_k]_{k\geq 1}$ where (X_k) is a CMD of E with* $\dim X_k < \infty$, *$k \geq 1$ and let q be the quotient map from L to L/E. Let Y be a finite dimensional subspace of L. For each n, let*

$Z_n \subseteq E$ be the finite dimensional subspace defined by $Z_n = [X_k]_{k=1}^n$. Then given any $0 < \varepsilon < 1$, there exists an integer s such that for any $y \in Y$, one can find $m \in Z_s$ satisfying

$$\|y + m\| \le (1+\varepsilon)\|qy\| + \varepsilon\|y\|.$$

Proof of Theorem 4.1: Let $(y_n) \subseteq S_{X^{**}}$ be w^*-null. We need to find a subsequence of (y_n) which is BCBS.

Denote by q the quotient map from X^{**} to X^{**}/X. We will choose the required subsequence of (y_n) in the following two steps.

STEP 1: We choose a subsequence (y_{k_i}) of (y_n) to satisfy the following property

(∗) Let $\gamma_i > 0$, $\gamma_i \downarrow$ be such that $\sum \gamma_i$ converges. Let (a_i) be a bounded sequence of reals. Suppose for $\varepsilon > 0$ and $C > 0$ we have $z_p = \sum_{i=n_{p-1}+1}^{n_p} a_i y_{k_i}$ satisfies, for all p, $C > \|z_p\| > \varepsilon$ and $\|qz\| \le \gamma_p 2^{-p}$. Then there exists $d \ge 1$ such that $\{z_p\}_{p=d}^{\infty}$ is a BCBS.

Since X has the (PC)-property it follows by Theorem 3.3, that there exists a boundedly complete SBFDD (X_i) for X. Let us denote $N_k = {[X_i]_{i\neq k}}^{\perp}$. Then each N_k is a finite dimensional subspace of X^* and so is $F_n = [N_k]_{k=1}^n$. Note that $F_n^{\top} = [X_i]_{i=n+1}^{\infty}$. We take $Z_n = [X_k]_{k=1}^n$.

By repeated use of Lemma 4.2 we find a subsequence (y_{k_n}) and two strictly increasing sequences $(s_n), (t_n)$ of natural numbers satisfying the following conditions:

$$\|y_{k_n}|_{F_{t_n+1}}\| < 2^{-n}\gamma_n; \tag{6}$$

$$\text{if } y \in Y_p = [y_{k_1}, \cdots, y_{k_p}], \text{ there exists } m \in Z_{s_p} \text{ such that } \|y+m\| < (1+\gamma_p)\|qy\| + \gamma_p\|y\|; \tag{7}$$

$$\text{for each } m \in Z_{s_p} \text{ there exists } \bar{m} \in [X_j]_{t_{p-1}+2}^{t_p} \text{ such that } \|m+\bar{m}\| < (1+\gamma_p)\text{dist}(m, [X_j]_{t_{p-1}+2}^{\infty}) + \gamma_p\|m\|. \tag{8}$$

Since $z_p \in Y_{n_p}$, by (7) there exists $m_p \in Z_{s_{n_p}}$ such that $\|z_p + m_p\| \le (1+\gamma_{n_p})\|qz_p\| + \gamma_{n_p}\|z_p\|$. Since $\|qz_p\| \le \gamma_p 2^{-p}$, we can find p large enough such that

$$\|z_p + m_p\| < \gamma_p. \tag{9}$$

By (6), for each r, $\|y_{k_{n_{p-1}+r}}|_{F_{t_{n_{p-1}+1}}}\| < 2^{-(n_{p-1}+r)}\gamma_{n_{p-1}+1}$. Hence,

$$\|z_p|_{F_{t_{n_{p-1}+1}}}\| \leq \sum_{i=n_{p-1}+1}^{n_p} |a_i|\|y_{k_i}|_{F_{t_{n_{p-1}+1}}}\|.$$

Since (a_i) is bounded, choosing p large enough, we can have $\|z_p|_{F_{t_{n_{p-1}+1}}}\| \leq \gamma_p$. By (9), it follows that $\|m_p|_{F_{t_{n_{p-1}+1}}}\| \leq 2\gamma_p$. This implies

$$\mathrm{dist}(m_p, [X_j]_{i=t_{n_{p-1}}+2}^{\infty}) < 2\gamma_p.$$

We now choose, by (8), $\bar{m_p} \in [X_j]_{i=t_{n_{p-1}}+2}^{t_{n_p}}$ such that

$$\begin{aligned}\|m_p + \bar{m_p}\| &< (1+\gamma_{n_p})\mathrm{dist}(m_p, [X_j]_{t_{n_{p-1}}+2}^{\infty}) + \gamma_{n_p}\|m_p\| \\ &< 2(1+\gamma_{n_p})\gamma_p + \gamma_{n_p}(C+\gamma_p) < (C+5)\gamma_p.\end{aligned}$$

Clearly, $(\bar{m_p})$ is a skipped block sequence of (X_j) and thus a BCBS. Since $\sum \gamma_p$ converges, there exists t such that $(m_p)_{p\geq t}$ is a BCBS. But $\|z_p + m_p\| \leq \gamma_p$ and hence there exists d such that $(z_p)_{p\geq d}$ is a BCBS.

Step 2: Let (y_{k_n}) be the subsequence obtained from Step 1. To simplify notation, we denote this subsequence by (y_k). Let us observe that $(*)$ holds for any subsequence of (y_k) (by putting some $a_i = 0$, if necessary).

Recall that (y_k) is a normalized w^*-null sequence in X^{**}. By a well-known result we can choose a subsequence, call it (y_k) again, such that (y_k) is basic, with basic constant C_1.

The following two cases can occur.

Case 1: Suppose for some subsequence (y_{k_m}), $\|qy_{k_m}\| \to 0$. Since X is a separable Banach space with the (PC)-property, it follows from [8] (see Theorem 3.3) that (y_{k_m}) has subsequence which is a BCBS.

Case 2: $\inf_k \|qy_k\| \geq \alpha$. Without loss of generality and passing to a subsequence if necessary, we assume that there exists $h \in X^\perp$, $\|h\| = 1$ and some $\alpha > 0$ such that $|h(y_k) - \alpha| < 2^{-k}$. Since X^{**}/X is reflexive, it follows that for some subsequence, which we denote by (y_k) again, there exists $z \in X^{**}$ $qy_k \xrightarrow{w} qz$.

Suppose (a_k) is such that $\sup_n \|\sum_1^n a_k y_k\| \leq M$ for some M. Then $\sup_n |h(\sum_1^n a_k y_k)| \leq M$ and since $|h(y_k) - \alpha| \leq 2^{-k}$, we get $\sup_n |\sum_1^n a_k| < \infty$. Hence there exists (n_k) such that $\sum_1^{n_k} a_k$ converges, to say, a.

We consider the following two cases.

Case (a): $\|qy_k - qz\| \to 0$.

In this case, it is straightforward to observe that $\sum_1^{n_k} a_k qy_k \to qu$ for some $u \in X^{**}$.

Case (b): $\|qy_k - qz\| > \varepsilon$ for some $\varepsilon > 0$. Since X^{**}/X is reflexive, we assume without loss of generality that $(q(y_k - z))$ is a BCBS.

Writing $\sum_{k=1}^{n_k} a_k qy_k = \sum_{k=1}^{n_k} a_k q(y_k - z) + (\sum_{k=1}^{n_k} a_k)qz$ and using the bounded completeness of $(q(y_k - z))$, it follows that $\sum_{k=1}^{n_k} a_k qy_k$ converges.

Thus in both cases, we obtain, for some suitable subsequence of (y_k), that there exists (n_k) such that $\sum_{k=1}^{n_k} a_k qy_k$ converges, whenever

$$\sup_n \|\sum_1^n a_k y_k\| \leq \infty.$$

We claim (y_k) is a BCBS.

To show this, observe that, since (y_k) is basic, it is enough to show that there exists (n_k) such that $\sum_{i=1}^{n_k} a_i y_i$ converges whenever $\sup_n \|\sum_1^n a_i y_i\| \leq M$ for some M. Thus assume on the contrary, that $\sum_{i=1}^{n_k} a_i y_i$ does not converge for any sequence (n_k). Note that by the consideration above, there exists (n_k) such that $\sum_{i=1}^{n_k} a_i q(y_i)$ converges. Let $\gamma_p > 0$ be such that $\sum \gamma_p < \infty$. Then there exists $\varepsilon > 0$ and a subsequence (n_{k_p}) such that $\|\sum_{n_{k_p}+1}^{n_{k_{p+1}}} a_i y_i\| \geq \varepsilon$ but $\|q(\sum_{n_{k_p}+1}^{n_{k_{p+1}}} a_i y_i)\| \leq \gamma_p 2^{-p}$. Observe that $\|\sum_{n_{k_p}+1}^{n_{k_{p+1}}} a_i y_i\| \leq 2M$. Since (y_k) is a basic sequence with basis constant C_1, it follows that $|a_i| < 4MC_1$ and hence is bounded.

Taking $z_p = \sum_{n_{k_p}+1}^{n_{k_{p+1}}} a_i y_i$ and $C = 2M$, it follows from $(*)$ that there exists d such that $(z)_{p \geq d}$ is a BCBS. However $\sup_n \|\sum_{p=d}^n z_p\| \leq 2M$. Thus $\sum_{p \geq d} z_p$ converges. This contradicts $\|z_p\| \geq \varepsilon$.

The proof is complete. ∎

Example 4.3: Consider the James tree space JT. It is a separable dual space with non-separable dual JT^*. Denoting the pre-dual of JT by B it is known that B has the (PC)-property (see [1]and [3]), and JT^*/B is isomorphic to $\ell_2(\Gamma)$ for some uncountable set Γ (see [16]). It follows from Theorem 4.1 that JT^* has property (s^*). Therefore the space JT^* shows that the property (s^*) is not enough for a dual space to be separable. Next let $A : l_2 \to JT$ be a one-to-one linear compact operator with dense range. Put $T_1 = A^* : JT^* \to l_2$. Since JT^* is not separable it follows from Proposition 3.1 that T_1 is not a G_δ-embedding. Since the notion of G_δ-embedding is separably defined, it follows that there is a separable subspace $E \subset JT^*$ such that the restriction $T = T_1|_E$ is not a G_δ-embedding. However, by using the (s^*) property of JT^* it is easy to see that any sequence $\{x_n\} \subset S_E$ with $\lim_n Tx_n = 0$ contains a subsequence which is BCBS. This shows that the tree hypothesis in Theorem 2.2 cannot be weakened to the sequence hypothesis.

Example 4.4: An example of a separable Banach space E with separable dual such that E fails the (PC)-property but has property (s) is the space B_∞ constructed in [8]. It is known that $JT_\infty = B_\infty^*$ is separable and B_∞ fails the (PC)-property (see [10, 11]). In the following proposition we prove B_∞ has property (s).

Proposition 4.5: *B_∞ has property (s).*

Proof. We will use the following properties of the space B_∞ (see [8, 9] for details).

There exists a sequence of Banach spaces (X_k), each isometric to ℓ_2 such that

(i) (X_k) is a (complemented) Schauder decomposition of B_∞.
(ii) (X_k) is a BCSBD of B_∞.

Let $P_n : X \to \sum_{k=1}^n \oplus X_k$ denote the projection on the first n components of $\sum_{k=1} \oplus X_k$. Since (X_k) is a Schauder decomposition, we have $\sup_n \|P_n\| < c$ for some $c > 0$.

Let $(x_i) \subseteq S_{B_\infty}$ be a w-null sequence. For each n, we have $(P_n x_i)_{i\geq 1}$ is weakly null. We consider the following two cases.

Case 1: For infinitely many n, $\lim_i \|P_n x_i\| = 0$. Choose n_1 and i_1 such that $\|P_{n_1} x_{i_1}\| < c^{-1}4^{-2}$. We can write $x_{i_1} = s_1 + t_1$ where $s_1 \in [X_k]_{k=1}^{n_1}$ and $t_1 \in [X_k]_{k=n_1+1}^{\infty}$, $\|s_1\| = \|x_{i_1} - t_1\| < c^{-1}4^{-2}$. There exists m_1 and $y_1 \in [X_k]_{k=n_1+1}^{m_1}$ such that $\|t_1 - y_1\| < c^{-1}4^{-1}$. Note that $\|x_{i_1} - y_1\| < c^{-1}4^{-1}$.

Choose $n_2 > m_1 + 1$ and i_2 such that $\|P_{n_2} x_{i_2}\| < c^{-1}4^{-3}$. We can write $x_{i_2} = s_2 + t_2$ where $s_2 \in [X_k]_{k=1}^{n_2}$ and $t_2 \in [X_k]_{k=n_2+1}^{\infty}$, $\|s_2\| = \|x_{i_2} - t_2\| < c^{-1}4^{-3}$. There exists m_2 and $y_2 \in [X_k]_{k=n_2+1}^{m_2}$ such that $\|t_2 - y_2\| < c^{-1}4^{-2}$. Thus $\|x_{i_2} - y_2\| < c^{-1}4^{-2}$.

Continuing, we obtain a subsequence (x_{i_k}) of (x_i) and $y_k \in [X_k]_{k=n_k+1}^{m_k}$, $n_k+1 \leq m_k < n_{k+1}$ such that $\sum_k^\infty \|x_{i_k} - y_k\| < 1/2c$. (y_k) is a skipped block sequence of (X_k), and hence it is a BCBS. By stability (x_{i_k}) is a BCBS.

Case 2: There exists m such that for all $n \geq m$, $\liminf_i \|P_n x_i\| > \delta$ for some δ. Without loss of generality, we assume for all n, $\lim_i \|P_n x_i\| > \delta$.

Since the range of each P_n is ℓ_2, by a standard diagonal argument we can find a subsequence of (x_i), which we denote by (x_i) again, such that for each n, $(P_n x_i)/\|P_n x_i\|$ is c_1 equivalent to the unit vector basis of ℓ_2.

We claim that (x_i) is BCBS. Otherwise, there exists a sequence (a_i) such that $\sup_m \|\sum_{i=1}^m a_i x_i\| < \infty$ but $\sum a_i x_i$ does not converge. Note that for each n, $\sup_m \|\sum_{i=1}^m a_i P_n x_i\| < \infty$ and hence $\sum a_i P_n x_i$ converges. Thus $\sum_i |a_i|^2$ converges.

Since $\sum a_i x_i$ does not converge, there exist an increasing sequence (n_p) and $\varepsilon > 0$ such that

$$\left\| \sum_{n_{p-1}+1}^{n_p} a_i x_i \right\| > \varepsilon$$

but

$$\left(\sum_{n_{p-1}+1}^{n_p} |a_i|^2 \right)^{1/2} \leq K 4^{-p}$$

where $k = 1/(cc_1)$.

Let $z_{p-1} = \sum_{n_{p-1}+1}^{n_p} a_i x_i$.

We have

$$\begin{aligned} \|P_1 z_1\| = \left\| \sum_{n_1+1}^{n_2} a_i P_1 x_i \right\| &\leq \left\| \sum_{n_1+1}^{n_2} a_i \frac{P_1 x_i}{\|P_1 x_i\|} \|P_1 x_i\| \right\| \\ &\leq cc_1 \left(\sum_{n_1+1}^{n_2} |a_i|^2 \right)^{1/2} \leq 4^{-1}. \end{aligned}$$

Thus there exists u_1, v_1 such that $z_1 = u_1 + v_1$ where $u_1 \in X_1$ and $v_1 \in [X_k]_{k\geq 2}$ with $\|u_1\| \leq 4^{-1}$.

We can find l_1 and y_1 such that $y_1 \in [X_k]_{k=2}^{l_1}$ and $\|v_1 - y_1\| \leq 4^{-1}$. Note that $\|z_1 - y_1\| < 2^{-1}$.

Consider P_{l_1+2}. Similar to above we have,

$$\|P_{l_1+2} z_2\| \leq cc_1 \left(\sum_{n_2+1}^{n_3} |a_i|^2 \right)^{1/2} \leq 4^{-2}.$$

We find u_2, v_2, y_2 and a number l_2 such that $z_2 = u_2 + v_2$, $u_2 \in [X_k]_{k=1}^{l_1+2}$, $\|u_2\| \leq 4^{-2}$ and $y_2 \in [X_k]_{k=l_1+3}^{l_2}$. We again have $\|z_2 - y_2\| < 2^{-2}$.

Continuing, we obtain a sequence of increasing integers l_p and vectors $y_p \in [X_k]_{l_{p-1}+3}^{l_p}$ such that $\|z_p - y_p\| < 2^{-p}$. But (y_p) is a skipped block sequence of (X_k) and, hence by (ii) is a BCBS. Thus there exists a $d \geq 1$ such that $(z_p)_{p\geq d}$ is a BCBS as well. Since for any k, $\sup_k \|\sum_{p=d}^{d+k} z_p\| < \infty$ it follows that $\sum_{p=d} z_p$ converges. This contradicts that $\|z_p\| > \varepsilon$ and the proof is complete. ∎

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Added in proof. Professor Rosenthal pointed out to us that he had observed before that in a Banach space with (PC)-property any semi-normalized basic sequence has a subsequence which is Boundedly complete. However, this was not published until recently in H. Rosenthal, Boundedly complete weak-Cauchy basic sequences in Banach space with the PCP, J. Func. Anal. 253 (2007), no. 2, 772–781. (See corollary 4 there). Since (PC)-prpoperty is a 3-space property, (see [19]) our Theorem 4.1 follows as a special case of his result.

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